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The Polarization of a QED Plasma in a Strong Magnetic Field

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Abstract

In this paper we study the polarization tensor of photons in a QED plasma in the presence of a magnetic field. We do it both at vanishing and at finite temperature. We use two different methods to compute the polarization tensor components $\Pi^{\mu\nu}$ within the one-loop approximation. The first starts from the effective Lagrangian of the system and relates $\Pi^{\mu\nu}$ to the thermodynamic quantities. The second makes use of the electron propagator in an external magnetic field. In this second approach we use an imaginary time formalism. These methods give consistent results in the first non-vanishing order in the photon 4-momentum. Beyond this limit the first method is not applicable and the introduction of the second can not be avoided. We give some physical interpretations of our results.

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1 Introduction

The investigation of the electromagnetic properties of a relativistic plasma in a strong external magnetic field is interesting both from a theoretical point of view and for possible applications in cosmology and astrophysics. Large magnetic fields are known to be present in neutron stars [1], where $B \sim 10^{12}$ G, in supernovas [2], where $B \sim 10^{14}$ G, and several models foresee very large magnetic fields in the early Universe [3]. When such strong magnetic fields are present the propagation of electromagnetic waves in a plasma is considerably modified with respect to the free field case. These strong fields can be not treated like a perturbation. Indeed, the tensorial structure of the polarization tensor need to be completely reconsidered [4].

According to the pioneering approach of Fradkin [5] one of our aims is to relate the electromagnetic properties of a QED plasma in a strong magnetic field to the thermodynamic properties of the system. This should give the reader a physically more transparent interpretation of the results that partially are present already in the literature, as well as of the new results that we obtain here.

All the thermodynamic properties of a QED plasma can be derived starting from the effective Lagrangian of the system. Assuming $\mathbf{B}(\mathbf{x}) = (0, 0, B)$ the Lagrangian can be written

$$\mathcal{L}^{\text{eff}} = \mathcal{L}^{\text{eff}}(B) + \mathcal{L}^{\text{eff}}(B, T, \mu), \quad (1)$$

where the vacuum contribution at the one loop level is [6]

$$\mathcal{L}^{\text{eff}}(B) = -\frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^3} \left[eBs \coth(eBs) - 1 - \frac{1}{3}(eBs)^2 \right] \exp(-m^2 s), \quad (2)$$

whereas the one-loop matter contribution is

$$\mathcal{L}^{\text{eff}}(B, \mu, T) = \frac{\ln Z}{\beta V} = \frac{1}{\beta} \frac{|e|B}{2\pi^2} \sum_{n=\bar{0}}^\infty \int_{-\infty}^\infty dk_3 \left\{ \ln[1 + e^{-\beta(E_n(k_3) - \mu)}] + \ln[1 + e^{-\beta(E_n(k_3) + \mu)}] \right\}. \quad (3)$$

Here

$$E_n(k_3) = \sqrt{k_3^2 + 2neB + m^2} \quad (4)$$

is the energy of the n -th Landau level at tree level. The sum $\sum_{n=0} \equiv \frac{1}{2} \sum_n (2 - \delta_{n0})$ takes into account the double spin degeneration of the Landau levels with the exception of the lowest. Equation (3) follows from a simple phase space consideration¹. In fact, due to the coalescing of the transverse momentum states into those of a two-dimensional harmonic oscillator, the number of available states for any given value of n is

$$\frac{V|e|B}{(2\pi)^2} (2 - \delta_{n0}) dk_3 \quad . \quad (5)$$

One of the most important effects of the modification of the electron phase-space is the “pair generation” that takes place when magnetic fields larger than $B_c \equiv m^2/e$ are applied to the plasma. We use quotation marks in order not to confuse the reader about the meaning of pair generation in this context. Clearly, a constant magnetic field can not induce pair generation from the vacuum. However, things can be different at finite temperature and/or density. Indeed, the number density of electrons+positrons is [7]

$$n = \frac{|e|B}{2\pi^2} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dk_3 \left\{ \frac{1}{1 + e^{\beta(E_n(k_3) - \mu)}} + \frac{1}{1 + e^{\beta(E_n(k_3) + \mu)}} \right\} \quad . \quad (6)$$

It increases roughly linearly with B when $B \gg B_c$ and $T \lesssim (|e|B)^{1/2}$. This phenomenon can be understood as a shift in the equilibrium of photons and pairs [8]: the equilibrium of the process $e^+e^- \leftrightarrow \gamma$ moves to the left owing to the growing of the number of available states that electrons and positrons can occupy in the lowest Landau level. Thanks to the amplification of its phase-space this level is practically the only occupied level when $eB \gg T^2$. Since B modifies the density of the charge carriers, it is reasonable to expect that the electromagnetic properties of the plasma have to be affected by strong magnetic fields. The charge neutrality of the plasma is preserved, provided that $\mu = 0$, since the increasing of the electron and positron energy densities balance each other. Indeed, the charge density is

$$\rho(\mu) = \frac{\partial \mathcal{L}^{\text{eff}}(B, T, \mu)}{\partial \mu} = \frac{|e|B}{2\pi^2} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dk_3 \left\{ \frac{1}{1 + e^{\beta(E_n(k_3) - \mu)}} - \frac{1}{1 + e^{\beta(E_n(k_3) + \mu)}} \right\} \quad . \quad (7)$$

¹ Although Eq.(3) can be obtained from the fermion propagator as done in refs.[8][9], we prefer to adopt here this simpler physical interpretation.

The magnetization plays an essential role in determining the screening properties of the plasma. Like the energy density, the magnetization \mathbf{M} can be obtained starting from the effective Lagrangian

$$\mathbf{M} = \frac{\partial \mathcal{L}}{\partial \mathbf{B}} = \frac{\partial \mathcal{L}^{\text{eff}}(B)}{\partial \mathbf{B}} + \frac{\partial \mathcal{L}^{\text{eff}}(B, T, \mu)}{\partial \mathbf{B}} \quad . \quad (8)$$

For example, the matter contribution to the magnetization in the limit $T = 0$ is (we refer the reader to ref. [9] for a more general expression)

$$M(B, 0, \mu) = \frac{|e|}{2\pi^2} \sum_{n=0}^{\left\lceil \frac{\mu^2 - m^2}{2eB} \right\rceil} \left(\mu \sqrt{\mu^2 - m^2 - 2eBn} - (m^2 + 4eBn) \log \frac{\mu + \sqrt{\mu^2 - m^2 - x}}{\sqrt{m^2 + x}} \right). \quad (9)$$

Note that in contrast to what happens for ρ , a component of the magnetization depends on the vacuum part of the effective Lagrangian. In other words, the vacuum polarizes in strong magnetic fields. The relative strength of the vacuum and matter polarization has been evaluated in ref.[9]. The vacuum contribution to the magnetization exceeds the matter contribution if $eB \gg T^2, |\mu^2 - m^2|$. We do not consider here the other vacuum contributions to the polarization tensor since they are already widely reported in the literature (see e.g. ref. [10]).

In the next section we will show that the components of the polarization tensor can be related to the thermodynamic quantities ρ , \mathbf{M} and the magnetic susceptibility $\chi = \partial M / \partial B$. In section 3 we write the general expression of the polarization tensor in terms of the fermion propagator in an external magnetic field. In section 4 we apply this expression to get the components of $\Pi^{\mu\nu}$ in the static limit and verify that they coincide with the results obtained in section 2. In section 5 we give examples of computations of some components of $\Pi^{\mu\nu}$ beyond the static limit. Finally, section 6 contains our conclusions.

2 The screening operator in terms of \mathcal{L}^{eff}

The polarization operator for a QED plasma is defined by

$$\Pi_{\mu\nu}(x, x') \equiv i \frac{\delta \langle j_\mu(x') \rangle}{\delta A_\nu(x)} = i \frac{\delta}{\delta A_\nu(x)} \frac{\delta \Gamma^{\text{eff}}}{\delta A_\mu(x')} \quad . \quad (10)$$

where $\Gamma^{\text{eff}} = \int d^4x \mathcal{L}^{\text{eff}}$ is the effective action. If the plasma has a non-vanishing chemical potential the tree level Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - e\cancel{A} - \gamma_0\mu - m)\psi \quad . \quad (11)$$

The form of Eq. (10) remains unchanged if we replace $A(x)$ with $\tilde{A}(x) \equiv (A^0 + \frac{1}{e}\mu, \mathbf{A})$. Note that only this combination of the chemical potential and the vector potential is physically meaningful [11]. In fact, whenever A_0 is changed, the polarization of the plasma rearranges so to fulfil $\mu + eA_0 = \text{const.}$ at the equilibrium [12]. Thus, μ is determined by the charge distribution of the plasma, and its value may depend on the position. It is then convenient to work in the gauge $A_0 = 0$ in order to have a well defined $\tilde{A}(x) = \mu = \text{const.}$ for the whole plasma.

As a first application of the above formula we compute the limit for $x \rightarrow x'$ of Π^{00} . We then have

$$\begin{aligned} \Pi^{00}(B, T, \mu) &= e^2 \frac{\partial^2 \mathcal{L}^{\text{eff}}(B, T, \mu)}{\partial \mu^2} = e^2 \frac{\partial^2 \rho}{\partial \mu^2} = \\ &= \frac{|e|^3 B \beta}{(2\pi)^2} \sum_{n=\hat{0}}^{\infty} \int_0^\infty dk_3 \left(\frac{1}{\cosh^2 \left(\frac{\beta(E_n(k_3) + \mu)}{2} \right)} + \frac{1}{\cosh^2 \left(\frac{\beta(E_n(k_3) - \mu)}{2} \right)} \right) \end{aligned} \quad (12)$$

where, assuming a uniform \mathbf{B} and working in the Euclidean space, we have used $\Gamma_{\text{mat}}^{\text{eff}} = -i\beta V \mathcal{L}(B, T, \mu)$.

It is worthwhile to compare this result with the corresponding quantity obtained for $B = 0$ [13]

$$\Pi^{00}(0, T, \mu) = \frac{e^2 \beta}{2(2\pi)^3} \int_0^\infty d^3\mathbf{k} \left(\frac{1}{\cosh^2 \left(\frac{\beta(E + \mu)}{2} \right)} + \frac{1}{\cosh^2 \left(\frac{\beta(E - \mu)}{2} \right)} \right) \quad (13)$$

Keeping in mind Eq.(5) it is evident at a glance that the effect of the magnetic field on Π^{00} is completely mediated by the modification of the electron phase

space. The increase of the screening power of static electric fields when overcritical magnetic fields are applied to the plasma can also be interpreted as due to the increase of the number of pairs (see Eq. (6)) as we anticipated in the introduction. In this sense the magnetic field plays a role analogous to that played by the temperature.

Let us now consider the Π^{0i} components ($i = 1, 2, 3$). In the same gauge as we used above, we have

$$\Pi_{0i}(x, x') = i \frac{\delta}{\delta \tilde{A}^0} \frac{\delta}{\delta \tilde{A}^i} \Gamma^{\text{eff}} = ie \varepsilon_{ijk} \partial_k \frac{\delta \rho}{\delta B^j(x')}. \quad (14)$$

where we used the chain rule for the functional derivative to get $\frac{\delta}{\delta \tilde{A}^i} = \varepsilon_{ij}^k \partial_k \frac{\delta}{\delta B^j}$. Whereas $\Pi_{03} = 0$ identically and $\Pi_{01} = \Pi_{02} = 0$ if the momentum insertion is zero or parallel to the field, the transversal components receive a non-vanishing contribution if the momentum insertion is perpendicular to the field. More precisely, the Fourier transform of (14) in terms of the photon momentum components p_i is

$$\Pi_{0i} = ie \varepsilon_{ij} p_i \frac{\partial \rho}{\partial B} + O(p^2) \quad i = 1, 2 \quad (15)$$

while $\Pi_{i0} = \Pi_{0i}^*$. These terms are related to the to the interaction of the net charge of the plasma with the magnetic field and contribute to the Hall conductivity. For the spatial components we get the general formula

$$\Pi_{ij} = \varepsilon_i^{kl} \varepsilon_j^{mn} p_k p_m \left[\left(\delta_{nl} - \frac{B_n B_l}{B^2} \right) \frac{1}{B} \frac{\partial}{\partial B} + \frac{B_n B_l}{B^2} \frac{\partial^2}{\partial B^2} \right] \mathcal{L}^{\text{eff}} \quad (16)$$

This gives

$$\Pi_{33} = p_2^2 \frac{1}{B} M \quad ; \quad (17)$$

$$\Pi_{11} = p_3^2 \frac{1}{B} M + p_2^2 \chi \quad ; \quad (18)$$

$$\Pi_{22} = p_3^2 \frac{1}{B} M + p_1^2 \chi \quad ; \quad (19)$$

$$\Pi_{12} = \Pi_{21} = p_1 p_2 \chi \quad ; \quad (20)$$

$$\Pi_{13} = \Pi_{31} = -p_3 p_1 \frac{1}{B} M \quad ; \quad (21)$$

$$\Pi_{23} = \Pi_{32} = p_3 p_2 \frac{1}{B} M \quad . \quad (22)$$

It is worthwhile to note the symmetry of the spatial part of the polarization tensor. This agrees with the tensorial structure obtained in ref.[4].

3 The electron propagator in an external magnetic field

Although the results of the previous section have been obtained without any use of the fermion propagator S , a treatment addressed to determine the components of the polarization tensor beyond the leading order in momenta can not leave S out of consideration. The fermion propagator in a constant magnetic field is given by [14]

$$S(x, x') = \sum_{n=0}^{\infty} \int \frac{d\omega dk_2 dk_3}{(2\pi)^3} e^{-i\omega(t-t') + ik_2(y-y') + ik_3(z-z')} \times \frac{1}{\omega^2 - k_3^2 - m^2 - 2eBn + i\epsilon} S(n; \omega, k_2, k_3) \quad (23)$$

where

$$S(n; \omega, k_2, k_3) = \begin{pmatrix} mI_{n,n} & 0 & -(\omega + k_3)I_{n,n} & -i\sqrt{2eBn}I_{n,n-1} \\ 0 & mI_{n-1,n-1} & i\sqrt{2eBn}I_{n-1,n} & -(\omega - k_3)I_{n-1,n-1} \\ -(\omega - k_3)I_{n,n} & i\sqrt{2eBn}I_{n,n-1} & mI_{n,n} & 0 \\ -i\sqrt{2eBn}I_{n-1,n} & -(\omega + k_3)I_{n-1,n-1} & 0 & mI_{n-1,n-1} \end{pmatrix} \quad (24)$$

and

$$I_{n,l} = I_{n;k_2}(x)I_{l;k_2}(x') \quad (25)$$

where

$$I_{n;k_2}(x) = \left(\frac{eB}{\pi}\right)^{1/4} e^{-\frac{1}{2}eB\left(x - \frac{k_2}{eB}\right)^2} \frac{1}{2^{n/2}\sqrt{n!}} H_n\left(\sqrt{eB}\left(x - \frac{k_2}{eB}\right)\right). \quad (26)$$

H_n are the Hermite polynomials and we have chosen the γ -matrices in the chiral representation. The propagator is obtained by solving Dirac's equation in a magnetic field. Some useful identities are

$$\sum_{n=0}^{\infty} I_{n;k_2}(x)I_{n;k_2}(x') = \delta(x - x') \quad (27)$$

and

$$\int_{-\infty}^{\infty} dx I_{n;k_2}(x) I_{l;k_2}(x) = \delta_{n,l}. \quad (28)$$

In terms of $S(x, x')$ the polarization tensor is

$$\Pi^{\mu\nu}(x) = e^2 \int dt' dx' dy' dz' \text{Tr} (\gamma^\mu S(x, x') \gamma^\nu S(x', x)). \quad (29)$$

Before we pass to the computation of the components of $\Pi^{\mu\nu}$ beyond the static limit we will show how the results of the previous section can be reproduced using Eq. (29).

4 The static limit

Whenever the momentum insertion from the electromagnetic field into the fermion loop is vanishing the polarization tensor can be written

$$\Pi^{\mu\nu} = \sum_{n,l=0}^{\infty} \int \frac{d\omega dk_2 dk_3}{(2\pi)^3} \frac{1}{(\omega^2 - k_3^2 - m^2 - 2eBn)(\omega^2 - k_3^2 - m^2 - 2eBl)} \pi^{\mu\nu} \quad (30)$$

where

$$\pi^{\mu\nu} = \int dx' \text{Tr} (\gamma^\mu S(n; \omega, k_2, k_3) \gamma^\nu S(l; \omega, k_2, k_3)) \quad (31)$$

It can be calculated, for example, that

$$\pi^{00} = 2(\omega^2 + k_3^2 + 2eBn + m^2) (I_{n,n} + I_{n-1,n-1}) \delta_{l,n} \quad (32)$$

$$\pi^{33} = 2(\omega^2 + k_3^2 - 2eBn - m^2) (I_{n,n} + I_{n-1,n-1}) \delta_{l,n} \quad (33)$$

$$\begin{aligned} \pi^{11} = & 2(\omega^2 - k_3^2 - m^2)(I_{n,n}\delta_{l,n+1} + I_{n-1,n-1}\delta_{l,n-1}) + \\ & 2eB \left(\sqrt{n(n+1)} I_{n-1,n+1} \delta_{l,n+1} + \sqrt{n(n-1)} I_{n,n-2} \delta_{l,n-1} \right) \end{aligned} \quad (34)$$

π^{11} differs from π^{22} only in that the last term has the opposite sign. As expected Π^{00} and Π^{33} correspond to zero angular momentum transfer between the photon and the plasma, implying the selection rule $\Delta n = 0$. Instead, transversally polarized waves can induce transitions between different Landau levels.

4.1 The $T = 0$ finite μ contribution

Let us consider the $T = 0$ contribution for the various components. We have

$$\Pi^{00}(B, 0, \mu) = \frac{8|e|^3 B}{(2\pi)^2} \sum_{n=0}^{\infty} \oint_C \frac{d\omega}{2\pi i} \int_0^{\infty} dk_3 \frac{\omega^2 + k_3^2 + 2eBn + m^2}{(\omega^2 - k_3^2 - 2eBn - m^2)^2} \quad (35)$$

To obtain this, perform the k_2 integration using (28). It follows that

$$\begin{aligned} \Pi^{00}(B, 0, \mu) &= \\ \frac{8|e|^3 B}{(2\pi)^2} \sum_{n=0}^{\infty} \oint \frac{d\omega}{2\pi i} \int_0^{\infty} dk_3 &\left(\frac{2\omega^2}{(\omega^2 - k_3^2 - 2eBn - m^2)^2} - \frac{1}{\omega^2 - k_3^2 - 2eBn - m^2} \right) = \\ \frac{8|e|^3 B}{(2\pi)^2} \sum_{n=0}^{\infty} \left(\frac{d}{dx} \int_0^{\sqrt{\mu^2 - m^2 - x}} dk_3 \sqrt{k_3^2 + m^2 + x} - \int_0^{\sqrt{\mu^2 - m^2 - x}} \frac{dk_3}{2\sqrt{k_3^2 + m^2 + x}} \right) &= \\ \frac{|e|^3 B}{\pi^2} \sum_{n=0}^{\left[\frac{\mu^2 - m^2}{2eB} \right]} \frac{\mu}{\sqrt{\mu^2 - m^2 - 2enB}}, &\quad (36) \end{aligned}$$

where $x = 2eBn$. As can be seen from above, the only non-vanishing contribution is coming from the point where the $\omega = E$ double pole is crossing the integration contour. More about the contour C and the methods we use to perform the integrals can be found in the Appendix.

There are two regions where the expression simplifies. One is the limit $B = 0$ where we get

$$\Pi^{00}(0, 0, \mu) = \frac{e^2 \mu \sqrt{\mu^2 - m^2}}{2\pi^2} \quad (37)$$

the other is $2eB > \mu^2 - m^2$ where

$$\Pi^{00}(B, 0, \mu) = \frac{|e|^3 B}{2\pi^2} \frac{\mu}{\sqrt{\mu^2 - m^2}} \quad (38)$$

Eq. (38) can be understood in terms of the modified relation between the charge density and the chemical potential of the plasma. In fact, in this limit Eq. (8) becomes

$$\rho(\mu) \approx \frac{eB}{2\pi^2} \sqrt{\mu^2 - m^2} \quad (39)$$

and Eq.(38) can easily be reproduced using $\Pi^{00} = e^2 \partial \rho / \partial \mu$. Eq. (37) disagrees with the result of refs.[13]. In fact, the authors obtain zero for $\Pi^{00}(0, 0, \mu)$. On the other hand, our Eq.(36) agrees completely with the result of ref.[15].

Let us now consider the spatial components and verify that they are zero. From (33) we have

$$\Pi^{33}(B, 0, \mu) = \frac{8|e|^3 B}{(2\pi)^2} \sum_{n=0}^{\infty} \oint \frac{d\omega}{2\pi i} \int_0^{\infty} dk_3 \frac{\omega^2 + k_3^2 - 2eBn - m^2}{(\omega^2 - k_3^2 - 2eBn - m^2)^2} = 0 \quad (40)$$

Again it is important to take the contour crossing into account in order to obtain the correct result.

$\Pi^{11}(B, 0, \mu)$ and $\Pi^{22}(B, 0, \mu)$ are simply computed. The last term of (34) is killed by the k_2 integration and we are left with a vanishing result. Hence we conclude that

$$\Pi^{11}(B, 0, \mu) = \Pi^{22}(B, 0, \mu) = \Pi^{33}(B, 0, \mu) = 0 \quad (41)$$

Thus static magnetic fields are not screened in the present limit. These equations clearly remain valid when $\mathbf{B} \rightarrow 0$. Again we disagree with the conclusion of ref.[13] where a magnetic screening length proportional to the Fermi momenta was obtained at $B_{\text{ext}} = 0$. The disagreement that we obtain here and for $\Pi^{00}(0, 0, \mu)$ suggests to us that the contributions from the crossing of the poles through the contour C was ignored by the authors of ref.[13].

Proceeding similarly to what we have done for the other components we easily deduce that $\Pi^{0j}(B, 0, \mu) = 0$ if there is no momentum insertion in the fermion loop.

4.2 The finite T contribution

At finite temperature we must add two more contour integrals:

$$\begin{aligned} \Pi^{00}(B, T, \mu) = & \frac{8|e|^3 B}{(2\pi)^2} \sum_{n=0}^{\infty} \left(\int_{-i\infty+\mu-\epsilon}^{i\infty+\mu-\epsilon} \frac{d\omega}{2\pi i} \frac{1}{1 + e^{-\beta(\omega-\mu)}} I(\omega, k, n) \right. \\ & \left. + \int_{-i\infty+\mu+\epsilon}^{i\infty+\mu+\epsilon} \frac{d\omega}{2\pi i} \frac{1}{1 + e^{\beta(\omega-\mu)}} I(\omega, k, n) + \oint_C \frac{d\omega}{2\pi i} I(\omega, k, n) \right) \end{aligned} \quad (42)$$

where

$$I(\omega, k, n) = \frac{\omega^2 + k^2 + 2eBn + m^2}{(\omega^2 - k^2 - 2eBn - m^2)^2} \quad . \quad (43)$$

The integrals are performed according to the prescription given in the Appendix with the result

$$\Pi^{00}(B, T, \mu) = \frac{|e|^3 B \beta}{4\pi^2} \sum_{n=0}^{\infty} \int_0^{\infty} dk_3 \left(\frac{1}{\cosh^2 \left(\frac{\beta(E+\mu)}{2} \right)} + \frac{1}{\cosh^2 \left(\frac{\beta(E-\mu)}{2} \right)} \right) \quad (44)$$

in agreement with (12). Although here the result coincides with that of ref.[13] in the limit $\mathbf{B} = 0$, this can easily be understood since the contributions from the pole crossing through the contours C and C_{\pm} cancel in this case. Similarly one can verify that the spatial components remain zero even at non-zero temperature. The behavior of Π^{00} as a function of B is shown in Fig. 1. As we anticipated in the introduction, we can see from this figure that the screening properties of the plasma are sensitive to the magnetic field only when $eB \gg T^2$. Thus the effect of even a large magnetic field can be neglected if this condition is not fulfilled.

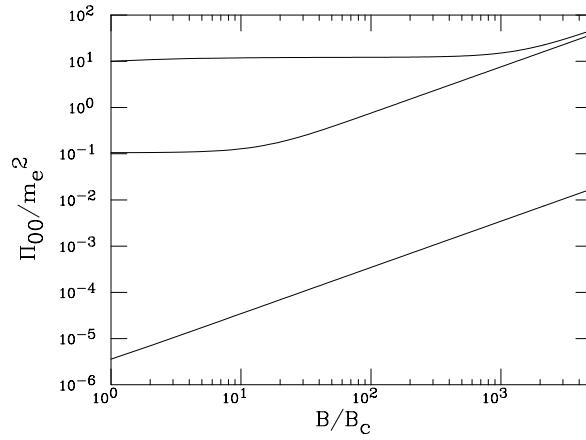


Fig.1. Π^{00} for three different temperatures. From top, $T = 10m_e$, $T = m_e$ and $T = 0.1m_e$.

5 Beyond the static limit

The application of an external magnetic field breaks the isotropy of the plasma. It is thus understandable that the electromagnetic response of the plasma will depend on the direction of the momentum insertion with respect to the field orientation. In this section we will give some sample calculations when the photon has non-zero momentum or energy.

5.1 $\mathbf{p} \parallel \mathbf{B}$

We start by considering a momentum insertion parallel to the magnetic field. Put $k'_3 = k_3 + p_3$ where p_3 is the momentum insertion. $\Pi^{\mu\nu}$ is then obtained by replacing k_3^2 with $k_3'^2$ in one of the propagators and replacing k_3^2 with $k_3 k'_3$ in $\pi^{\mu\nu}$ in Eqs.(32-34). To compute Π^{00} we use

$$I(\omega, k_3, n) = \frac{\omega^2 + k_3 k'_3 + 2enB + m^2}{(\omega^2 - E_n(k_3)^2)(\omega^2 - E_n(k'_3)^2)} \quad (45)$$

as the integrand in Eq.(35). Performing the ω integration over the contour C (see the appendix) this becomes

$$\frac{m^2 + 2eBn}{2p_3} \left(\int_{-\sqrt{\mu^2 - m^2 - 2eBn}}^{\sqrt{\mu^2 - m^2 - 2eBn}} \frac{dk_3}{(2k + p_3)\sqrt{k^2 + m^2 + 2eBn}} - (p_3 \rightarrow -p_3) \right) \quad (46)$$

Expanding to second order in p_3^2 we find

$$\begin{aligned} \Pi^{00}(B, 0, \mu|p_3) &= \Pi^{00}(B, 0, \mu|p_3 = 0) + \\ &\frac{p_3^2 |e|^3 B}{\pi^2} \sum_{n=\hat{0}}^{\left[\frac{\mu^2 - m^2}{2eB}\right]} \left(\frac{\mu}{12(\mu^2 - m^2 - 2eBn)^{3/2}} - \frac{\mu}{6\sqrt{\mu^2 - m^2 - 2eBn}(m^2 + 2eBn)} \right) \\ &\quad + O(p_3^4) \end{aligned} \quad (47)$$

where $\Pi^{00}(B, 0, \mu)$ is given by Eq.(36).

At $T \neq 0$ we have to manage two simple poles in the complex ω -plane. As discussed in the Appendix, the integration over C_+ splits in two integrals each of them corresponding to the two possible positions of the positive-energy pole: $E > \mu$ or $E < \mu$. A third integral, over C_- , receives a contribution only

from the negative-energy pole, always contained in C_- . An T -independent term coming from the integration over C_+ and C_- cancel exactly with the result of the integral over C given by Eq.(46). Thus we get

$$\Pi^{00}(B, T, \mu|p_3) = -\frac{8|e|^3 B}{(2\pi)^2} \sum_{n=0}^{\infty} \frac{2eBn + m^2}{p_3} \times \int_{-\infty}^{\infty} \frac{dk}{(2k + p_3)\sqrt{k^2 + 2eBn + m^2}} \left(\frac{1}{1 + e^{\beta(\omega - \mu)}} + \frac{1}{1 + e^{\beta(\omega + \mu)}} \right). \quad (48)$$

Previously we found that $\Pi^{33}(B, T, \mu) = 0$ at zero momentum. At non-zero p_3 we have to integrate

$$\frac{\omega^2 + k_3 k'_z - 2enB - m^2}{(\omega^2 - E_n(k_3)^2)(\omega^2 - E_n(k'_z)^2)} \quad (49)$$

over k_3 and ω . The residue at $\omega = E_n$ is $-\frac{k_3}{2E_n p_3}$. Clearly, $\Pi^{33} = 0$ even at non-zero p_3 after integrating over both signs of k_3 . This agrees with Eq.(17).

It is also worthwhile to check Π^{11} with non-zero p_3 . Using Eq.(34) we find that we have now to manage the integrand

$$\frac{\omega^2 - k_3 k'_3 - m^2}{(\omega^2 - k_3^2 - m^2 - 2neB)(\omega^2 - k_3'^2 - m^2 - 2(n+1)eBl)}. \quad (50)$$

After the integrations we get

$$\Pi^{11}(B, 0, \mu|p_3) = \frac{p_3^2}{B} \left[\frac{|e|}{2\pi^2} \sum_{n=0}^{\infty} \left(\mu \sqrt{\mu^2 - m^2 - 2eBn} - (m^2 + 4eBn) \log \frac{\mu + \sqrt{\mu^2 - m^2 - x}}{\sqrt{m^2 + x}} \right) \right] + 0(p_3^4) \quad (51)$$

that agree with Eq.(18) as the reader can check keeping in mind Eq.(9). The same result as (51) is found for $\Pi^{22}(B, 0, \mu|p_3)$.

Let us now move on to Π^{0j} . Using (31) we find that

$$\pi^{03} = 2\omega (k_3 + k'_3) (I_{n,n} + I_{n-1,n-1}) \delta_{n,l} \quad (52)$$

This clearly gives zero even for $p_3 \neq 0$. In fact, the integrand can be rewritten as

$$\frac{\omega}{p_3} \left(\frac{1}{\omega^2 - k^2 - 2eBn - m^2} - \frac{1}{\omega^2 - k'^2 - 2eBn - m^2} \right) \quad (53)$$

Shifting the k' in the second term gives zero after integration. We leave to verify $\Pi_{01}(B, 0, \mu|p_3) = \Pi_{02}(B, 0, \mu|p_3) = 0$ as an easy exercise for the reader.

5.2 $\mathbf{p} \perp \mathbf{B}$

The case with momentum transverse to the magnetic field is slightly more complicated. Of special interest are the Π^{0i} ($i = 1, 2$) components at non-zero p_i which give through

$$\sigma^{ij} = i \frac{\partial \Pi^{0i}(p)}{\partial p_j} \quad (54)$$

the plasma conductivity. Using (31) we find that

$$\pi^{01} = \int dx' 4i\sqrt{2eB}l\omega (I_{n,n} + I_{n-1,n-1}) I_{l,l-1} \quad (55)$$

At non-zero p_2 we have to calculate

$$\begin{aligned} \int dx' dy' dk_2 dk'_2 e^{ik_2(y-y') - ik'_2(y-y') + ip_2(y-y')} I_{n;k_2}(x) I_{n';k_2}(x') I_{l';k'_2}(x') I_{l;k'_2}(x) = \\ eB \int dx dx' I_{n;0}(x) I_{n';0}(x') I_{l;p_2}(x) I_{l';p_2}(x') \end{aligned} \quad (56)$$

This can be found to be

$$eB e^{-\frac{p_2^2}{2eB}} \left(-\frac{p_2}{(2eB)^{1/2}} \right)^{n+n'-l-l'} \left(\frac{l!l'!}{n!n'!} \right)^{1/2} L_l^{n-l} \left(\frac{p_2^2}{2eB} \right) L_{l'}^{n'-l'} \left(\frac{p_2^2}{2eB} \right) \quad (57)$$

for $n \geq l$ and $n' \geq l'$, where the L 's are Laguerre-polynomials. Expanding to order p_2 we find, in our case,

$$- \sqrt{2eB} p_2 (l \delta_{n,l+1} + l \delta_{n,l}) + (n \rightarrow n+1) \quad (58)$$

and to order p_2^3

$$\frac{p_2^3}{2\sqrt{2eB}} \left((3l^2 + l) \delta_{n,l+1} - (l^2 + l) \delta_{n,l+2} + (3l^2 - l) \delta_{n,l} - (l^2 - l) \delta_{n,l-1} \right)$$

$$+ (n \rightarrow n + 1). \quad (59)$$

For larger powers of p_2 , larger jumps between the Landau levels are allowed. We get

$$\begin{aligned} \Pi^{01}(B, 0, \mu|p_2) &= \frac{ie^2 p_2}{\pi^2} \sum_{n=0}^{\left[\frac{\mu^2-m^2}{2eB}\right]} \left(\sqrt{\mu^2 - m^2 - 2eBn} - \frac{eBn}{\sqrt{\mu^2 - m^2 - 2eBn}} \right) \\ &\quad - \frac{iep_2^3}{4\pi^2 B} \sum_{n=1}^{\left[\frac{\mu^2-m^2}{2eB}\right]} \left(2n\sqrt{\mu^2 - m^2 - 2eBn} - \frac{3eBn^2}{\sqrt{\mu^2 - m^2 - 2eBn}} \right) + O(p_2^5). \end{aligned} \quad (60)$$

Π_{02} is given by the same expression with p_2 replaced by p_1 . Eq.(60) confirms the result for the conductivity of [15] and extends it to second order in momentum. Note that the higher order correction vanishes for $2eB > \mu^2 - m^2$. The expression for the conductivity given in ref.[16], on the other hand, do not agree with our result or the result of [15].

We are also able to extend our result to finite temperature.

$$\begin{aligned} \Pi^{01}(B, T, \mu|p_2) &= \frac{ie^2 p_2}{\pi^2} \sum_{n=0}^{\infty} \left(1 + B \frac{d}{dB} \right) \int_0^{\infty} dk_3 \left(\frac{1}{1 + e^{\beta(E-\mu)}} - \frac{1}{1 + e^{\beta(E+\mu)}} \right) \\ &\quad - \frac{iep_2^3}{4\pi^2 B} \sum_{n=1}^{\infty} \left(2n + 3Bn \frac{d}{dB} \right) \int_0^{\infty} dk_3 \left(\frac{1}{1 + e^{\beta(E-\mu)}} - \frac{1}{1 + e^{\beta(E+\mu)}} \right) \\ &\quad + O(p_2^5) \end{aligned} \quad (61)$$

The first order term can again be checked using (15).

We may also consider Π^{00} and Π^{ii} at nonzero p_1 and p_2 . The calculation uses the same method as above. After expanding the Laguerre-polynomials and performing the contour-integrals, we can verify Eqs.(17-19). On the other hand we find

$$\begin{aligned} \Pi^{00}(B, 0, \mu|p_2) &= \Pi^{00}(B, 0, \mu, p_2 = 0) + \\ &\quad \frac{e^2 p_2^2}{2\pi^2 eB} \sum_{n=0}^{\infty} \left(\mu\sqrt{\mu^2 - m^2 - 2eBn} + m^2 \log \frac{\mu + \sqrt{\mu^2 - m^2 - x}}{\sqrt{m^2 + 2eBn}} - \frac{4eBn\mu}{\sqrt{\mu^2 - m^2 - 2eBn}} \right) \\ &\quad + O(p_2^4) \end{aligned} \quad (62)$$

Together with (47) we then have the complete expression to second order in momentum. In this case we can not check the result using section 1.

5.3 Energy insertion

Let us now consider the case in which the photon energy is different from zero. In this case the energy of one of the fermions in the polarization loop will be shifted to $\omega' = \omega + p_0$ where the photon energy p_0 is assumed to be real. To compute $\Pi^{00}(B, 0, \mu)$, the integrand to replace the one in Eq.(35) is

$$I(\omega, k_3, n) = \frac{\omega\omega' + k_3^2 + 2enB + m^2}{(\omega^2 - E_n(k_3)^2)(\omega'^2 - E_n(k_3)^2)} \quad . \quad (63)$$

Performing the ω -integration we find

$$\begin{aligned} \Pi^{00}(B, 0, \mu|p_0) = & \frac{|e|^3 B}{p_0 \pi^2} \left(\sum_{n=\hat{0}}^{\left[\frac{\mu^2 - m^2}{2eB}\right]} \sqrt{\mu^2 - 2enB - m^2} - \sum_{n=\hat{0}}^{\left[\frac{(\mu - p_0)^2 - m^2}{2eB}\right]} \sqrt{(\mu - p_0)^2 - 2enB - m^2} \right) = \\ & \Pi^{00}(B, 0, \mu) + \frac{2|e|^3 B}{\pi^2} \sum_{n=\hat{0}}^{\left[\frac{\mu^2 - m^2}{2eB}\right]} \left(p_0 \frac{2enB + m^2}{2(\mu^2 - 2enB - m^2)^{3/2}} + p_0^2 \frac{\mu(2enB + m^2)}{(\mu^2 - 2enB - m^2)^{5/2}} \right) \\ & + O(p_0^3) \quad . \quad (64) \end{aligned}$$

At $T \neq 0$ we have instead

$$\begin{aligned} \Pi^{00}(B, T, \mu|p_0) = & \frac{8|e|^3 B}{(2\pi)^2 p_0} \sum_{n=\hat{0}}^{\infty} \left[\int_{-\infty}^{\infty} dk_3 \left(\frac{1}{e^{\beta(E-\mu)} + 1} - \frac{1}{e^{\beta(E+\mu)} + 1} \right) \right. \\ & \left. - \int_{-\infty}^{\infty} dk_3 \left(\frac{1}{e^{\beta(E-(\mu+p_0))} + 1} - \frac{1}{e^{\beta(E+(\mu+p_0))} + 1} \right) \right] \quad (65) \end{aligned}$$

6 Conclusions

In this paper we have studied the polarization tensor of a QED plasma placed in a magnetic field. Although this has been the subject of several studies, some discrepancies, even in the simplest limit $B = 0$ [13], called for a more careful analysis. Furthermore, recent results concerning the thermodynamics of a QED plasma in a strong magnetic field allow a new, physically more

transparent, interpretation of old results as well as of the new results that we obtain here.

We have used two different methods. The first relate the polarization tensor directly to the thermodynamic quantities of the system. The second makes use of the fermion propagator in an external magnetic field. These methods give the same results in the first non-vanishing order in the photon 4-momentum. Beyond this limit the first method is not applicable and the introduction of the second can not be avoided. For instance, this is needed if one wants to find the dispersion relations for the electromagnetic waves.

Some of our results have a simple physical interpretation. Static electric fields are screened by the plasma in strong magnetic fields ($eB > T^2, \mu^2 - m^2$) more effectively than in the free field case. This can be understood since static electric fields are screened by the charge rearrangement in the plasma. At $T = 0$ no thermal pair production is active and the screening can be achieved only in the presence of a charge asymmetry. In strong magnetic fields the relation with the charge asymmetry is modified. Indeed, the charge density becomes proportional to the magnetic induction when eB is larger than the Fermi momentum squared, $\mu^2 - m^2$ (see Eq.(39)). Consequently, Π^{00} grows linearly with B (see Eq.(38)).

At $T \neq 0$, as we discussed in the introduction, the thermal pair production is amplified by large magnetic fields. Then more charge carriers are available to screen static electric fields. The reader can see in Fig. 1 that Π^{00} starts to grow linearly in B , once all electrons and positrons have dropped in the lowest Landau level. In this static limit the anisotropy induced by the external magnetic field can not play any role and the definition of an electric screening length $\lambda^{-2} = m_{el}^2 = \Pi^{00}(p_0 = 0, \mathbf{p} \rightarrow 0)$ [12] is still meaningful.

Static magnetic fields are not screened at all. In fact, macroscopic spatial currents can not be obtained if $\mathbf{p} = 0$ (see Eq.16). This conclusion can not be modified in presence of strong external magnetic fields neither at vanishing nor at finite temperature. However, magnetic screening is achieved if $\mathbf{p} \neq 0$ and Eqs.(17-22) show how this is related to the magnetization and magnetic susceptibility of the plasma in a non trivial way.

We also computed the electric conductivity of the plasma. Our result confirms that of ref.[15], but we improve the calculation by including second order terms in the momentum expansion. On the other hand, we disagree with the result of ref.[16].

The determination of the complete dispersion relations for electromagne-

tic waves propagating through plasmas in strong magnetic fields is beyond the purposes of the present paper. One would need to take into account the full tensorial structure of the polarization operator as done in Refs.[4]. Using our results according to the prescription of Refs.[4], would provide physically more accessible information about the propagation of electromagnetic waves through a plasma in presence of strong magnetic fields.

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Appendix

Let us give some details on how the contours are specified. At finite temperature the integral $\int d\omega$ is replaced by a sum over $\omega = \frac{i\pi}{\beta}(2n+1)$. This can be accomplished by integration along a counter clockwise contour around the poles of $\tanh \frac{\pi z}{2}$, e.g.

$$\sum_n f(2n+1) = \frac{i}{4} \oint dz \tanh \frac{\pi z}{2} f(z) \quad (66)$$

Deforming the contours gives

$$\begin{aligned} & \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} f(\omega) \int_{-i\infty+\mu+\epsilon}^{i\infty+\mu+\epsilon} \frac{d\omega}{2\pi i} f(\omega) \frac{1}{e^{\beta(\omega-\mu)} + 1} + \\ & \int_{-i\infty+\mu-\epsilon}^{i\infty+\mu-\epsilon} \frac{d\omega}{2\pi i} f(\omega) \frac{1}{e^{-\beta(\omega-\mu)} + 1} + \oint_C \frac{d\omega}{2\pi i} f(\omega) \end{aligned} \quad (67)$$

The first term is the divergent vacuum contribution, the second and third terms are non-zero only at finite temperature and the last term is the $T = 0$ non-zero μ contribution. The contours are depicted in fig. 2.

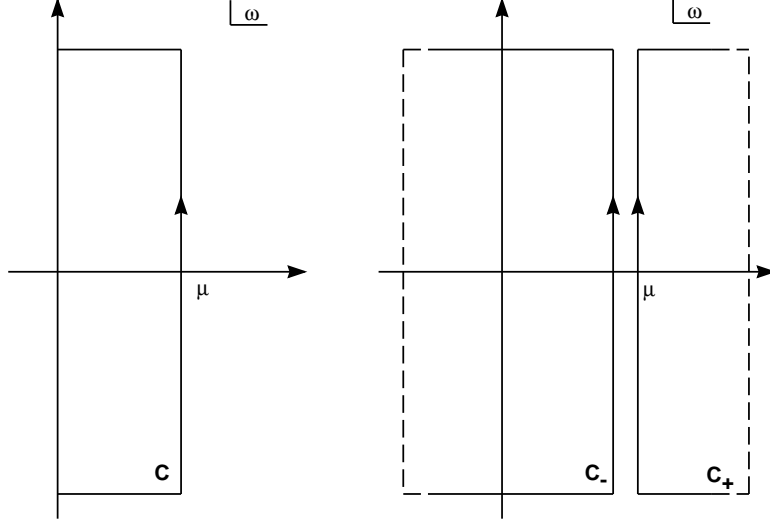


Fig. 2. Contours for $T = 0$ on the left and for $T \neq 0$ on the right.

It is worthwhile also to give some details about how we performed the integrals. Mainly we wish to direct the attention of the reader to the crossing of poles through the contours of integration.

We start by considering the $T = 0$ finite-density contribution. This term has the form

$$\oint_C \frac{d\omega}{2\pi i} \int_{-\infty}^{\infty} dk f(\omega, k) = \sum_i \lim_{\omega \rightarrow -E_i} \frac{d^{m_i-1}}{d\omega^{m_i-1}} \int_{-\infty}^{\infty} dk f(\omega, k) (\omega - E_i)_i^m \theta(\mu - \omega) \quad (68)$$

where we have assumed that the function $f(\omega, k)$ has poles $\omega = E_i$ on the real axis of order m_i . The contour of integration C is depicted in Fig. 2. The result of the integration is zero if the poles lie outside the region delimited by the contour C . This justifies the θ -function on the left side of Eq.(68). The effect of the θ -function is simply to modify the effective integration limits in the case the pole is first-order.

More interesting, when the pole is second-order (we do not come through higher order poles in our one-loop computations) a new term appears containing a delta-function coming from the derivative of the θ . This term corresponds to the crossing of the pole through the contour of integration. The δ kills the k -integral and leaves us with a term of the form like that in the result of Eq.(36).

The finite temperature contribution has the general form

$$\begin{aligned} & \int_{-i\infty+\mu+\epsilon}^{i\infty+\mu+\epsilon} \frac{d\omega}{2\pi i} \int_{-\infty}^{\infty} dk f(\omega, k) \frac{1}{e^{\beta(\omega-\mu)} + 1} + \\ & \int_{-i\infty+\mu-\epsilon}^{i\infty+\mu-\epsilon} \frac{d\omega}{2\pi i} \int_{-\infty}^{\infty} dk f(\omega, k) \frac{1}{e^{-\beta(\omega-\mu)} + 1} + \oint_C \frac{d\omega}{2\pi i} \int_{-\infty}^{\infty} dk f(\omega, k) \end{aligned} \quad (69)$$

When the poles are first order the first two, temperature dependent, integrals split in three non-vanishing parts. The first part corresponds to the integral over the contour C_+ (see Fig. 2.) for values of k such that the pole $\omega = E$ is inside C_+ ($E > \mu$). It can be written

$$\int_{-\infty}^{\infty} dk f(E, k) \frac{1}{e^{\beta(E-\mu)} + 1} - \int_{-\sqrt{\mu^2-x-m^2}}^{\sqrt{\mu^2-x-m^2}} dk f(E, k) \frac{1}{e^{\beta(E-\mu)} + 1} \quad ; \quad (70)$$

the second part corresponds to the integral over C_- when only the pole $\omega = -E$ is inside C_-

$$\int_{-\infty}^{\infty} dk f(-E, k) \frac{1}{e^{\beta(E+\mu)} + 1} \quad (71)$$

and the third one comes from the contribution to the same integral when k is such that the pole $\omega = E$ is inside C_-

$$\int_{-\sqrt{\mu^2-x-m^2}}^{\sqrt{\mu^2-x-m^2}} dk f(E, k) \frac{1}{e^{-\beta(E-\mu)} + 1} \quad . \quad (72)$$

It is then straightforward to verify that the sum of the second integral in (70) and the integral in (72) is temperature independent and cancels with the integral over C in (70). Then (70) takes the form

$$\int_{-\infty}^{\infty} dk f(E, k) \left(\frac{1}{e^{\beta(E-\mu)} + 1} + \frac{1}{e^{\beta(E+\mu)} + 1} \right) \quad (73)$$

where the separate contributions of electrons and positrons are evident. If the poles are second-order we have to take care of the contribution coming from the derivative of the θ -function integrated through the C_{\pm} contours. This generates the terms

$$\begin{aligned} & \int_{-\infty}^{\infty} dk f(E, k) \frac{1}{e^{\beta(E-\mu)} + 1} \delta(E - \mu) + \int_{-\infty}^{\infty} dk f(E, k) \frac{1}{e^{-\beta(E-\mu)} + 1} \delta(E - \mu) = \\ & \frac{1}{2} \left(\frac{dE}{dk} \right)_{E=\mu}^{-1} f(E = \mu) = \frac{\mu}{\sqrt{\mu^2 - x - m^2}} f(E = \mu) \quad . \end{aligned} \quad (74)$$

This is a T independent term. When inserted in (70) it cancel the result of the integral over C .

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